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Generalized Thomson Problem for 5 points

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Abstract

Optimal configurations on the sphere have various applications in physics, chemistry, biology, and computer science. The Generalized Thomson Problem is to place $N$ points on the unit sphere so that their energy (sums of all weighted reciprocal distances) will be minimized. This paper investigates the case for $N=5$. We show that among all configurations with two antipodal points the best one is the triangular bi-pyramid. Computational comparison with the square pyramid is provided and open problems discussed.

Background

The Thomson problem is to determine the minimum energy configuration of $N$ electrons on the surface of a sphere that repel each other with a force given by Coulomb's law. The physicist J. J. Thomson posed the problem in 1904 and later related problems and generalizations were posed later and were great interest not just for mathematicians, also for physicists and other scientists.

Here is the mathematical statement of the problem. Let $x_1, x_2, \ldots, x_N$ be a collection of $N$ distinct points on the unit sphere centered at the origin. The energy of this configuration is defined to be $E = \sum_{i<j} \frac{1}{|x_i - x_j|}$. Thomson problem is to minimize this energy over all possible configurations of $N$ distinct points on the unit sphere.

The Generalized Thomson problem defines the energy to be $E = \sum_{i<j} f(|x_i - x_j|)$ where $f(r) = \frac{1}{r^s}$ for all integer values of $s$ greater than zero. Tammes problem for $s = \infty$, Thomson problem for $s = 1$, and Whyte's problem for $s = 0$ are the notable cases of this generalized problem. Also configurations of $N$ points on a sphere with higher dimensions is considered to be a research interest for scientists.

Case for 5 points

While the best configurations for cases with 2, 3, 4, 6, 12 points is known, for $N = 5$ the best configuration is not known. However, based on various computations, everyone agrees that triangular bi-pyramid (TBP) is the optimal configuration. In the triangular bi-pyramid two points are antipodal while the remaining 3 points are the vertices of an equilateral triangle on the equator between the two poles.

In our case we assume that two of the points are antipodal, and we will prove that in this case the other 3 points form an equilateral triangle on the equator. Also, numerical computations will be given to compare the energy for triangular bi-pyramid and for square pyramid.

Lemma 1

Let $A$ and $B$ be endpoints of the diameter of a circle. $Y$ be a point on the circumference such that $AY = YB$ and $X$ be any random point on the circumference such that points $A$, $X$, and $B$ form a triangle. Then energy of triangle $AYB$ is always less or equal to the energy of $EXB$.

**Proof:** We start by using an equation which is greater or equal to zero and we will get the results at the end of the statements.

$$|x^2 - 2xy| = 0 \Rightarrow |x^3 - 4xy^2 + 4y^3| \geq 0$$

Adding up the last two inequalities:

$$E(ABC) \geq E(AXYB)$$

**Lemma 2**

Let $A$ and $B$ be the two antipodal points and $X, Y, Z$ any three points on the sphere. Let points $X, Y$ and $Z$ form a triangle with a circumscribed circle with radius $r$, where $(R = \text{radius of the sphere})$. Construct a similar triangle to $XYZ$ with points $x, y, z$ on the equator. Then $E(AYZB) \leq E(AYXZB)$.

**Lemma 3**

Among all triangles inscribed in a given circle, the equilateral one has the largest area.

**Proof:** Let $A$, $B$, and $C$ be the three points on the circle. Let $\theta$ be the angle between $AB$ and $AD$ where $AD$ is any diameter of the circle. The area of the triangle is $\frac{1}{2} \cdot AB \cdot AD \sin \theta = \frac{1}{2} \cdot AB \cdot BC \sin \theta$. To maximize this area, we have $\sin \theta = 1$, therefore $\theta = \frac{\pi}{2}$, which makes the triangle equilateral.

**Lemma 4**

Among all inscribed triangles in a given circle, energy of the equilateral triangle is less or equal to the energy of any other triangle.

**Proof:** Let $ABC$ be the equilateral triangle and $XYZ$ be a random triangle with sides $x, y, z$ inscribed in a circle with radius $R$. Then $AB = AC = IC = \sqrt{3}R \sin(\theta)$. Area($ABC$) = $\frac{1}{2} \cdot \sqrt{3}R^2 \sin(\theta)$. Area($XYZ$) = $\frac{1}{2} \cdot x \cdot R \sin(\theta)$. So, $E(ABC) = E(XYZ)$.

**Proof**

Here is the proof that triangular Bi-pyramid is the best configuration for 5 points with two of them being antipodal.

Let $A$ and $B$ be the north and south poles of the sphere, respectively. From Lemma 2 we get that in order for the energy to be minimum the other 3 points must be on the equator of the sphere. From Lemma 4 we get that the energy of the triangle on the equator is minimum when the triangle is equilateral, and since the distance between poles and vertices of the triangle is fixed, the general energy of the figure will be minimized as well.