Gödel’s Second Incompleteness Theorem

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Gödel’s Incompleteness Theorems

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Tutorial III:
Gödel’s Second Incompleteness Theorem

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Apologies

- Today’s session was meant to be the highlight, presenting tons of exciting stuff.
- Then a former student of mine told me that I’m going way too fast and losing listeners left and right.
- I therefore changed today’s presentation and hope I can go slow enough for the benefit of all who are still present.
- Apologies to those I lost earlier but also apologies to those who will find it too pedestrian today.
Section I: Introduction
Disclaimer, well-known by now . . .

- ‘G1’ is short for “Gödel’s First Incompleteness Theorem,” i.e., the incompletablility of arithmetic.
- ‘G2’ is short for “Gödel’s Second Incompleteness Theorem,” i.e., the unprovability of consistency (to be made more precise today).

There are different ways to establish various incompleteness results for specific formal systems using specific methods, which do not, however, necessarily transfer or generalize. (For today’s topic, this means in particular set theoretic proofs for G2.)

By contrast, we here understand both G1 and G2 as proofs (or methods of proof) that provide a uniform method applicable to a wide range of formal systems resulting in optimal results. G1 and G2 “scale,” so to speak.
## Today’s goals

<table>
<thead>
<tr>
<th>Goal 1</th>
<th>Understand the general proof strategy for G2 (while not all details).</th>
</tr>
</thead>
<tbody>
<tr>
<td>Goal 2</td>
<td>Learn about a few refinements.</td>
</tr>
<tr>
<td>Goal 3</td>
<td>Gain an appreciation of the scope of G2.</td>
</tr>
</tbody>
</table>
Section II: A First Proof
Stating G₂

Theorem

Let \( \mathcal{F} \) be consistent formal system. If \( \mathcal{F} \) satisfies the four derivability conditions listed below, then it cannot formally prove a certain formulation \( \text{CON}_F \) of its own consistency; in short:

\[ \not\vdash_\mathcal{F} \text{CON}_F. \]

(DC1) \( \vdash_\mathcal{F} \varphi \iff \vdash_\mathcal{F} \text{Pr}_F(\ulcorner \varphi \urcorner), \text{ all } \varphi \in \mathcal{L}_F \)

(DC2) \( \vdash_\mathcal{F} \text{Pr}_F(\ulcorner \varphi \urcorner) \land \text{Pr}_F(\ulcorner \varphi \rightarrow \psi \urcorner) \rightarrow \text{Pr}_F(\ulcorner \psi \urcorner), \text{ all } \varphi, \psi \in \mathcal{L}_F \)

(DC3) \( \vdash_\mathcal{F} \text{Pr}_F(\ulcorner \varphi \urcorner) \rightarrow \text{Pr}_F(\ulcorner \text{Pr}_F(\ulcorner \varphi \urcorner) \urcorner), \text{ all } \varphi \in \mathcal{L}_F \)

(DC4) \( \vdash_\mathcal{F} \gamma \leftrightarrow \neg \text{Pr}_F(\ulcorner \gamma \urcorner), \text{ for at least one } \gamma \in \mathcal{L}_F \)
The newcomers

- (DC2) $\vdash F \text{Pr}_F(\neg \varphi) \land \text{Pr}_F(\neg \varphi \to \psi) \to \text{Pr}_F(\neg \psi)$
  - DC2: provable closure under modus ponens (detachment)
  - Equivalent to $\vdash F \text{Pr}_F(\neg \varphi) \to \text{Pr}_F(\neg \varphi)$ $\to \text{Pr}_F(\neg \psi)$
  - By DC1+2, if $\vdash \varphi \to \psi$, then $\vdash \forall F(\neg \varphi) \to \text{Pr}_F(\neg \psi)$

- (DC3) $\vdash F \text{Pr}_F(\neg \varphi) \to \text{Pr}_F(\neg \text{Pr}_F(\neg \varphi))$
  - DC3 follows from provable $\Sigma_1$-completeness:
    $\vdash \varphi \to \text{Pr}_F(\neg \varphi)$, for all $\varphi \in \Sigma_1$
    since $\text{Pr}_F(\neg \varphi) \in \Sigma_1$.

- Clutter reduction act: Write “□$\varphi$” instead of “$\text{Pr}_F(\neg \varphi)$”
The proof idea

1. According to G1 (first half) we have:
   
   If $\mathcal{F}$ is consistent, then $\not\vdash_\mathcal{F} \gamma$.

2. Formalize that part of the proof in $\mathcal{F}$:
   
   $\vdash_\mathcal{F} \text{CON}_\mathcal{F} \rightarrow \neg \Box \gamma$.

3. Hence, by the fixed point equivalence $\vdash_\mathcal{F} \gamma \leftrightarrow \neg \Box \gamma$:
   
   $\vdash_\mathcal{F} \text{CON}_\mathcal{F} \rightarrow \gamma$.

4. Thus, $\not\vdash_\mathcal{F} \text{CON}_\mathcal{F}$, for otherwise $\vdash_\mathcal{F} \gamma$, contrary to G1.

   (This proof sketch was Gödel’s original proof!)
Preliminaries I

- Let \( \perp \) be either any logical absurdity (say, of the form ‘\( p \land \neg p \)’), or, in systems of arithmetic, any refutable falsehood (e.g., \( 0 = 1 \), or \( 0 \neq 0 \)).

- Observation. \( F \) is consistent iff \( \not\vdash F \perp \neq \perp \).

- Proof (indirect). First assume \( F \) is consistent and \( \vdash F \perp \neq \perp \). Since all our systems prove basic truths, they also prove \( \vdash F \perp \neq \perp \); hence, \( F \) is inconsistent contrary to the assumption. Thus, the consistency of \( F \) entails \( \not\vdash F \perp \neq \perp \). Now assume that \( \not\vdash F \perp \neq \perp \) and \( F \) is inconsistent. But an inconsistent system proves anything, so \( \vdash F \perp \neq \perp \) contrary to the assumption, Thus, \( \not\vdash F \perp \neq \perp \) entails the consistency of \( F \).

- “\( \neg \Box \perp \)” (the unprovability of \( \perp \)) can thus serve as a formalized consistency statement.
Preliminaries II

- Recall that $\neg \varphi$ is equivalent to $\varphi \rightarrow \bot$.
- $\Box \neg \varphi$ thus entails $\Box (\varphi \rightarrow \bot)$ and, by DC2, $\Box \varphi \rightarrow \Box \bot$.
- Hence, from $\Box \varphi \rightarrow \Box \neg \varphi$ we get $\Box \varphi \rightarrow (\Box \varphi \rightarrow \Box \bot)$.
- We can simplify the latter to $\Box \varphi \rightarrow \Box \bot$ since $p \rightarrow (p \rightarrow q) \vdash (p \rightarrow q)$ by propositional logic.
- Taking it together:

\[ (*) \quad \vdash_F \Box \varphi \rightarrow \Box \neg \varphi \quad \Rightarrow \quad \vdash_F \Box \varphi \rightarrow \Box \bot \]
Our first proof

(1) \( \vdash \varphi \gamma \leftrightarrow \neg \Box \gamma \); DC4
(2) \( \vdash \varphi \Box \gamma \rightarrow \neg \gamma \); PL(1)
(3) \( \vdash \varphi \Box \Box \gamma \rightarrow \Box \neg \gamma \); DC1+2(2)
(4) \( \vdash \varphi \Box \gamma \rightarrow \Box \Box \gamma \); DC3
(5) \( \vdash \varphi \Box \gamma \rightarrow \Box \neg \gamma \); PL(3,4)
(6) \( \vdash \varphi \Box \gamma \rightarrow \Box \bot \); (\(\star\))(5)
(7) \( \vdash \varphi \neg \Box \bot \rightarrow \neg \Box \gamma \); PL(6)
(8) \( \vdash \varphi \text{CON} \rightarrow \gamma \); CON \(\equiv \neg \Box \bot \), DC4(7)
(9) \( \not\vdash \varphi \text{CON} \); G1: \( \not\vdash \varphi \gamma \)
Tangent: A fixed point calculation

- Observe:
  1. $\vdash F \bot \rightarrow \gamma$; PL
  2. $\vdash F \square \bot \rightarrow \square \gamma$; DC1+2(1)
  3. $\vdash F \neg \square \gamma \rightarrow \neg \square \bot$; PL(3)

- Combine with,
  4. $\vdash F \neg \square \bot \rightarrow \neg \square \gamma$; see (7) above

- to get,
  5. $\vdash F \neg \square \bot \leftrightarrow \neg \square \gamma$; PL(3,4)
  6. $\vdash F \text{CON} \leftrightarrow \neg \square \gamma$; CON $\equiv \neg \square \bot$

- In other words, CON is a fixed point $\gamma$ s.t.: $\gamma \leftrightarrow \neg \Pr_F(\neg \gamma)$.
The next step(s)

▶ In order to complete the proof, we need to establish
\[(\text{DC2}) \vdash \Box \varphi \land \Box (\varphi \rightarrow \psi). \rightarrow \Box \psi,\]
and
\[(\text{DC3}) \vdash \Box \varphi \rightarrow \Box \Box \varphi,\]
for suitable formal systems.

▶ Bernays (1939) was the first (and last, actually) to give rigorous proofs for DC’s similar to DC2+3; their current formulation is due mostly to Löb (1955).
  ▶ Gödel never formally proved G2, nor did he establish any DC’s. It’s not unlikely, however, that he told Bernays about his plans during a common voyage across the Atlantic.
Section III:
The 2nd Derivability Condition
The task

- Prove

\[(DC2) \vdash_{\mathcal{F}} \text{Pr}_F(\Gamma \varphi) \land \text{Pr}_F(\Gamma \varphi \rightarrow \psi) \rightarrow \text{Pr}_F(\Gamma \psi)\]

- Assuming that $\mathcal{F}$ is closed under modus ponens, we need to show that $\mathcal{F}$ can formally prove that fact.

- Formally proving something like this means proving certain facts about Gödel numbers in $\mathcal{F}$.

- Let’s have a brief glimpse at what this means . . .
Preliminaries I

- Let $s$ be a sequence number, i.e.,
  $$s = 2^{e_1} \cdot 3^{e_2} \cdot 5^{e_3} \cdot \ldots \cdot \wp^{e_n};$$
  then let $(s)_k$ be the $k$-th exponent of $s$, i.e., $(s)_k = e_k$.

- Assume our coding uses prefix notation, i.e.,
  $$gn(\varphi \to \psi) = 2^{g(\rightarrow)} \cdot 3^{gn(\varphi)} \cdot 5^{gn(\psi)}.$$

- Then, if $n = gn(\varphi \to \psi)$, we have
  $$(n)_2 = gn(\varphi) \text{ and } (n)_3 = gn(\psi).$$
Preliminaries II

Let $\text{Expn}(x)$, $\text{Impl}(x)$, and $x \ast y$ be defined as follows:

- $\text{Expn}(n)$ is true iff $n$ is the Gödel number of an expression
- $\text{Impl}(n)$ is true iff $n = \equiv$ is the Gödel number of an implication
- $x \ast y = z$ iff $x, y, z$ are Gödel numbers for the sequences $\Gamma, \Gamma', \Delta$ resp., and $\Delta$ is the concatenation of $\Gamma$ with $\Gamma'$, i.e.: $\Gamma \bowtie \Gamma' \equiv \Delta$. 

What to prove I

\[ \vdash_{\mathcal{F}} \text{Fmla}(u) \land \text{Impl}(v) \land u = (v)_2 \to. \]
\[ [\text{Proof}_F(x, u) \land \text{Proof}_F(y, v) \to. \text{Proof}_F(x \otimes y \otimes \varphi_k^{(v)_3}, (v)_3)]. \]

Explanation. Suppose \( u, v \) satisfy the first antecedent; i.e., \( u = (v)_2 = gn(\varphi) \) and \( v = gn(\varphi \to \psi) \). Assume further that the encoded expressions are derivable in \( \mathcal{F} \); i.e., there is a \( \mathcal{F} \)-derivation \( d_1 \vdash_{\mathcal{F}} \varphi \) with \( x = gn(d_1) \) and a \( \mathcal{F} \)-derivation \( d_2 \vdash_{\mathcal{F}} \varphi \to \psi \) with \( y = gn(d_2) \). Then concatenate \( d_1 \) with \( d_2 \) and add one application of modus ponens to get a new \( \mathcal{F} \)-derivation \( d := d_1 \overset{\mathcal{F}}{\bullet} d_2 \overset{\bullet}{\bullet} \psi \) such that \( \text{Proof}_F(gn(d), gn(\psi)) \) holds. But since \( gn(d) = x \ast y \ast \varphi_k^{(v)_3} \), with \( k = lh(x + y) + 1 \) (i.e., the next prime number in the sequence), and \( gn(\psi) = (v)_3 \), we have to show that \( \mathcal{F} \) proves: \( \text{Proof}_F(x \circledast y \circledast \varphi_k^{(v)_3}, (v)_3) \), where ‘\( \circledast \)’ represents ‘\( \ast \)’ in \( \mathcal{F} \).
Abbreviations

- Let $z = x \otimes y \otimes \phi_{k}^{(v)3}$, and let $A(u, v, x, y)$ be short for:
  \[(\text{Fmla}(u) \land \text{Impl}(v) \land u = (v)_2) \land (\text{Proof}_F(x, u) \land \text{Proof}_F(y, v))\]

- We can thus write: $\vdash_F A(u, v, x, y) \rightarrow \text{Proof}_F(z, (v)_{3})$

- Now let us zoom in on $\text{Proof}_F(z, (v)_{3})$.

  - Disclaimer. Details depend of course on the chosen coding and the formal system $F$. 
What to prove II

- Proof\(F(z, (v)_3)\) will be somewhat of the following form:

\[
Seq(z) \land (v)_3 = (z)_{lh(z)} \land \forall k \leq lh(z)\left[ \ldots \right],
\]
whose first part reads \(z\) is a sequence number and its last exponent encodes the expression whose Gödel number is \((v)_3 = gn(\psi)\).

- The second big part expands to:

\[
\forall k \leq lh(z) \left[ Axiom_F((z)_k) \lor \exists i, j < k[Mp((z)_i, (z)_j, (z)_k) \lor \exists i < k[Gen((z)_i, (z)_k)]] \right],
\]
which reads: and all exponents in \(z\) encode either an axiom of \(F\) or, there are earlier exponents in \(z\), such that, taken together, they encode a valid instance of modus ponens or of universal generalization.

- All this now has to be formally derived from \(A(u, v, x, y)\).
What to prove

- The ugly truth: Because the proof is so messy, no one ever bothered to write it up.
  - Well, except for Bernays (1939).

- Excuse: Because most of the reasoning involves primitive recursive functions and formal systems under consideration are complete with respect to those, we don’t really need to bother.
  - Note quite correct, IMHO, see Buldt (1997).
Section IV:
The 3rd Derivability Condition
What to prove

- DC3 follows from provable $\Sigma_1$-completeness:
  \[
  \vdash \varphi \rightarrow \Pr_F(\neg \varphi), \text{ for all } \varphi \in \Sigma_1,
  \]
  since $\Pr_F(\neg \varphi) \in \Sigma_1$.

- Strategy: To get provable $\Sigma_1$-completeness from provable $\Delta_0$-completeness.
Definitions

- Recap. $\Delta_0$ includes all expressions in the language of arithmetic $\{0; S; +, \cdot\}$ without unbounded quantifiers.
  - Terms are exclusively composed from $\{0, S, +, \cdot\}$; e.g., $SS0$, $SS0 + S0$, or $S0 \cdot SS0$.
  - The only atomic expressions are equality statements among such terms; e.g., $S0 + S0 = S0 \cdot SS0$. Molecular expressions are formed as usual.
  - Quantification bounded by a numeral is included, since these can be rewritten as quantifier-free finite conjunctions, or disjunctions, resp.
  - Note that details may differ.

- A $\Sigma_1$-expression is a $\Delta_0$-expression prefixed with one unbounded existential quantifier.
\[ \Delta_0\text{-completeness} \]

**Theorem**

*Any extension \( \mathcal{F} \) of \( \mathcal{Q} \) is \( \Delta_0 \)-complete.*

**Proof.**

*By induction on the built-up of expressions.*
**Σ₁-completeness**

**Theorem**

*Any extension $\mathcal{F}$ of $\mathcal{Q}$ is $\Sigma_1$-complete.*

**Proof.**

Let $\exists x \varphi(x)$ be $\Sigma_1$ and true. Hence, there is number $n \in \mathbb{N}$, such that $\varphi(\bar{n})$ is true and, by $\Delta_0$-completeness, provable in $\mathcal{F}$: $\vdash_{\mathcal{F}} \varphi(\bar{n})$. But then one application of existential introduction proves $\vdash_{\mathcal{F}} \exists x \varphi(x)$. □
\( \Delta_0 \)-soundness

**Theorem**

Any extension \( \mathcal{F} \) of \( Q \) is \( \Delta_0 \)-sound if consistent.

**Proof.**

Assume otherwise, i.e., there is an expression \( \delta(n) \) that is false but \( \mathcal{F} \) proves it: \( \vdash_{\mathcal{F}} \delta(n) \). Since \( \delta(n) \) is false, \( \neg \delta(n) \) must be true and hence provable since \( \mathcal{F} \) is \( \Delta_0 \)-complete: \( \vdash_{\mathcal{F}} \neg \delta(n) \). Hence, \( \mathcal{F} \) is inconsistent, which is a contradiction.
Provable $\Delta_0$-completeness

**Theorem**

Any extension $\mathcal{F}$ of $Q + \text{IND}$ proves its own $\Delta_0$-completeness; in short:

$$\vdash_{\mathcal{F}} \delta \rightarrow \text{Pr}_F(\Gamma \delta \downarrow), \text{ for all } \delta \in \Delta_0$$

**Proof.**

By induction on the built-up of expressions, but induction now performed inside $\mathcal{F}$.

- Induction for quantifier-free expression, $I\Delta_0$, yields the above result.
- Induction for $\Sigma_1$-expression, $I\Sigma_1$, yields the above result but allows for free variables:

$$\vdash_{\mathcal{F}} \delta(x) \rightarrow \text{Pr}_F(\Gamma \delta(x) \downarrow).$$
**Provable $\Sigma_1$-completeness**

**Theorem**

Any extension $\mathcal{F}$ of $Q + I \Sigma_1$ proves its own $\Sigma_1$-completeness; in short:

$$\vdash_{\mathcal{F}} \varphi \rightarrow \text{Pr}_{\mathcal{F}}(\neg \varphi), \text{ for all } \varphi \in \Sigma_1$$

**Proof.**

1. $\vdash_{\mathcal{F}} \delta(x) \rightarrow \text{Pr}_{\mathcal{F}}(\neg \delta(x))$; prov. $\Delta_0$-completeness
2. $\vdash_{\mathcal{F}} \delta(x) \rightarrow \exists x \delta(x)$; QL
3. $\vdash_{\mathcal{F}} \text{Pr}_{\mathcal{F}}(\neg \delta(x)) \rightarrow \text{Pr}_{\mathcal{F}}(\neg \exists x \delta(x))$; DC1+2(2)
4. $\vdash_{\mathcal{F}} \delta(x) \rightarrow \text{Pr}_{\mathcal{F}}(\neg \exists x \delta(x))$; PL(1,3)
5. $\vdash_{\mathcal{F}} \exists x \delta(x) \rightarrow \text{Pr}_{\mathcal{F}}(\neg \exists x \delta(x))$; $\exists$-Intro
The million $$$ question

- How difficult is the proof for
  \[ \vdash \neg \delta(x) \rightarrow \text{Pr}_F(\neg \delta(x) \land) \] ?

- Difficult enough that no one except for Bernays (1939) published a rigorous and sufficiently detailed sketch.
  - Here, I willfully ignore Boolos (1993) since he more or less skipped the hard part, the induction basis.
Summary

- The informal reasoning for how to prove both DC2 and DC3 is sound and compelling.

- The tedious nature of those proofs, however, and their demand for continued attention to tiny and subtle details makes them a highly unrewarding undertaking. Everyone is happy to accept them on good faith.

- Be that as it may, this completes our first proof of G2.
Section V: A few refinements
Löb’s Theorem

Theorem

Assume that \( \mathcal{F} \) satisfies DC1–4. Then: \( \vdash_{\mathcal{F}} \Box \varphi \rightarrow \varphi \) iff \( \vdash_{\mathcal{F}} \varphi \)

Proof. (note that \( \Leftarrow \) is trivial (weakening))

(1) \( \vdash_{\mathcal{F}} \gamma \leftrightarrow (\Box \gamma \rightarrow \varphi) \) ; DC4
(2) \( \vdash_{\mathcal{F}} \Box \gamma \rightarrow (\Box \Box \gamma \rightarrow \Box \varphi) \) ; PL, DC1+2(1)
(3) \( \vdash_{\mathcal{F}} \Box \gamma \rightarrow \Box \Box \gamma \) ; DC3
(4) \( \vdash_{\mathcal{F}} \Box \gamma \rightarrow \Box \varphi \) ; PL(2,3)
(5) \( \vdash_{\mathcal{F}} \Box \varphi \rightarrow \varphi \) ; assumption
(6) \( \vdash_{\mathcal{F}} \Box \gamma \rightarrow \varphi \) ; PL(4,5)
(7) \( \vdash_{\mathcal{F}} \gamma \) ; PL(6,1)
(8) \( \vdash_{\mathcal{F}} \Box \gamma \) ; DC1(7)
(9) \( \vdash_{\mathcal{F}} \varphi \) ; PL(6,8)
Löb’s Theorem

- It’s okay if the proof for LT feels “weird,” for it is.
- A formalized version FLT is available: \( \vdash \neg F \square \bot \rightarrow \bot \); PL
- FLT is the single most important axiom to make the modal study of provability (“provability logic”) work.

LT entails G2:

- (1) \( \vdash \neg \square \bot \); PL
- (2) \( \vdash \neg \square \bot \rightarrow \bot \); LT(1)
- (3) \( \vdash \neg \square \bot \); PL(2)
Interpretability

- We write $U \triangleleft V$ if theory $V$ interprets theory $U$.
  - Recap of the main idea. A translation $\tau$ among two languages is such that it maps variable (constant) symbols to variable (constant) symbols and commutes with non-logical symbols and logical operators. Then $V$ interprets $U$ if there is a translation $\tau$ such that: $(\ast) \vdash_U \varphi \Rightarrow \vdash_V \tau(\varphi)$, for all sentences $\varphi \in \mathcal{L}_{U,0}$.

- Note that if $U \triangleleft V$, then, if $V$ is consistent, so must be $U$.
  - Proof (indirect). $\vdash_U \varphi \land \neg \varphi$ entails $\vdash_V \tau[\varphi \land \neg \varphi]$; hence $\vdash_V \tau[\varphi] \land \tau[\neg \varphi]$, and thus $\vdash_V \tau[\varphi] \land \neg \tau[\varphi]$, rendering $V$ inconsistent contrary to the assumption.

- The following two theorems are due to Feferman (1960).
Two refinements of G2

- **Theorem:** Assume $\mathcal{F}$ to extend $\mathcal{PA}$. Then $F + \text{CON}_F \not\vdash F$.
  - $\mathcal{F}$ cannot prove its own consistency; no, it cannot even find it in itself to interpret it.

- **Theorem:** Assume $\mathcal{F}$ to extend $\mathcal{PA}$. Then $F + \neg \text{CON}_F \not\vdash F$.
  - $\mathcal{F}$ cannot prove its own consistency; no, it even accounts for its own inconsistency.

- An crucial role in these proofs play “Rosser--esque” provability predicates.
G2 for weak systems

- Weak systems such as $Q$ were never in contention for G2; too weak to "express" (whatever that means) consistency.
  - Consequently, model-theoretic proofs that establish G2 for $Q$ were dismissed.

- Then definable cuts and related techniques changed the perception.

- Let $Q \subseteq T$; then there is a consistency statement $CON_T$ s.t.: $\not\vdash_T CON_T$.

- The proof, due to Pudlak (1985), works with the simultaneous use of two provability predicates (an idea due to Jan Mycielski) and generalizes to consistency statements over arbitrary cuts (roughly: sets defined by open expressions that are closed under ‘≤’ and ‘S’). Related research about formal systems that do not recognize certain functions as total has been conducted by Dan Willard. These results vastly expand the scope of G2.
Section VI: The scope of G2
Intensionality

▷ G1 is extensional, *viz.*, G1 is stable under substitution of provability predicates.
  ▷ We can prove G1 using Gödel’s provability predicate or Rosser’s version (see yesterday’s slides).

▷ G2 might be intensional, *viz.*, G2 might not be stable under substitution of provability predicates.
  ▷ We mentioned that the Rosser predicate has the following property: $\vdash \mathcal{F} \neg \varphi \Rightarrow \vdash \mathcal{F} \neg \text{Pr}_F^R(\neg \varphi \neg)$.
  ▷ If $\mathcal{F}$ is consistent: $\vdash \mathcal{F} \neg \bot$; but then, by the Rosser property: $\vdash \mathcal{F} \neg \text{Pr}_F^R(\neg \bot \neg)$. Thus, $\mathcal{F}$ proves its own (Rosser-)consistency.
Intensionality?

▶ The question of G2’s alleged intensionality has not yet received a satisfactory and widely accepted answer.

▶ As such, the scope of G2, and whether G2 really enforces limitations on consistency proofs, are contended issues.

▶ More generally: While G1 has found with recursion theory its proper place in the grand scheme of things (see yesterday’s lecture), scholars do not yet agree on what G2’s proper place should be.
Section VII: Summary
Today’s goals

Goal 1

Understand the general proof strategy for G2 (while not all details).

Summary

1. We explained the reasoning for why DC2 and DC3 are true and provable.

2. We highlighted, however, also the tedious nature of their fully formalized proofs.
Today’s goals

Goal 2
Learn about a few refinements.

Summary

1. Streamlining G2 via Löb’s Theorem.
2. Strengthening G2 via interpretability.
3. Extending G2 to weak systems via cuts.
Today’s goals

Goal 3

Gain an appreciation of the scope of G2.

Summary

1. Mentioned, embarrassingly brief, the (alleged?) intensionality of G2.
Introduction

A First Proof

The Second DC

The Third DC

A few refinements

Scope of G2

Select Literature


