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On the Riesz energy of spherical designs

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Abstract. We show how polynomial techniques can be applied for obtaining upper and lower bounds on the Riesz energy of spherical designs.

1 Introduction

A spherical \(\tau\)-design \(C \subset S^{n-1}\) is a finite nonempty subset of \(S^{n-1}\) such that

\[
\frac{1}{\mu(S^{n-1})} \int_{S^{n-1}} f(x) d\mu(x) = \frac{1}{|C|} \sum_{x \in C} f(x)
\]  

(1)

(\(\mu(x)\) is the Lebesgue measure) holds for all polynomials \(f(x) = f(x_1, x_2, \ldots, x_n)\) of degree at most \(\tau\). The number \(\tau = \tau(C)\) is called strength of \(C\).

The spherical designs were introduced in 1977 by Delsarte-Goethals-Seidel [5] where the authors proved that the minimum possible cardinality of a \(\tau\)-design on \(S^{n-1}\) is at least

\[
D(n, \tau) = \begin{cases} 
2\binom{n+e-2}{n-1}, & \text{if } \tau = 2e - 1, \\
\binom{n+e-1}{n-1} + \binom{n+e-2}{n-1}, & \text{if } \tau = 2e.
\end{cases}
\]

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Let $\alpha > 0$, $C \subset S^{n-1}$ be a spherical $\tau$-design and

$$W(C, n, \alpha) = \sum_{x, y \in C, x \neq y} [d(x, y)]^{-2\alpha} = \sum_{x, y \in C, x \neq y} [2(1 - \langle x, y \rangle)]^{-\alpha}$$

be the Riesz energy of $C$. Denote by

$$W(N, n, \tau, \alpha) = \inf \{ W(C, n, \alpha) : |C| = N, C \subset S^{n-1}, C \text{ is } \tau\text{-design} \}$$

the minimum possible $\alpha$-energy of spherical $\tau$-designs on $S^{n-1}$ of $N$ points. Denote also $h(t) = [2(1 - t)]^{-\alpha}$.

Configurations with minimal or near minimal Riesz energy have been a source of many investigations (see [1], [4], [6], [11]). In this regard estimations on the quantity $W(N, n, \tau, \alpha)$ are important. Energy of designs on $S^2$ were considered in [7, 8].

In this note we show how polynomial techniques can be applied for obtaining lower and upper bounds on $W(N, n, \tau, \alpha)$ and give some examples for small $\tau$.

2 Some preliminaries

For fixed dimension $n$, the Gegenbauer polynomials [12] are defined by $P_0^{(n)} = 1$, $P_1^{(n)} = t$ and the three-term recurrence relation

$$(k + n - 2)P_{k+1}^{(n)}(t) = (2k + n - 2)tP_k^{(n)}(t) - kP_{k-1}^{(n)}(t) \text{ for } k \geq 1.$$  

We note that $\{P_i^{(n)}(t)\}$ are orthogonal in $[-1, 1]$ with a weight $(1 - t^2)^{(n-3)/2}$. If $f(t) \in \mathbb{R}[t]$ is a real polynomial of degree $k$ then $f(t)$ can be uniquely expanded in terms of the Gegenbauer polynomials as $f(t) = \sum_{i=0}^{k} f_i P_i^{(n)}(t)$. The coefficients $f_i$, $i = 0, 1, \ldots, k$, are important in the so-called linear programming theorems. The identity

$$|C|f(1) + \sum_{x, y \in C, x \neq y} f(\langle x, y \rangle) = |C|^2 f_0 + \sum_{i=1}^{k} \frac{f_i}{r_i} \sum_{j=1}^{r_i} \left( \sum_{x \in C} v_{ij}(x) \right)^2$$  \hspace{1cm} (2)

is an important source of estimations by polynomial techniques. Here $C \subset S^{n-1}$ is a spherical code, $f(t) = \sum_{i=0}^{k} f_i P_i^{(n)}(t)$ as above, $\{v_{ij}(x) : j = 1, 2, \ldots, r_i\}$ is an orthonormal basis of the space Harm($i$) of homogeneous harmonic polynomials of degree $i$ and $r_i = \dim \text{Harm}(i)$. In the classical case (cf. [5, 9]) the sums of the both sides are neglected for suitable polynomials.

The identity (2) can be used for estimations of the Riesz energy of spherical codes (see [1]). Here we show that either lower and upper bound are possible when the code is a $\tau$-design with positive $\tau$. 

3 General bounds

The general frame of the linear programming bounds on $W(N, n, \tau, \alpha)$ is given by the next two theorems.

An equivalent definition of spherical designs says that $\sum_{x \in C} v_{ij}(x) = 0$ for every $i \leq \tau$ and every $j \leq r_i$ (cf. [9]). This suggests that polynomials of degree at most $\tau$ could be useful – the right hand side of (2) is then reduced to $|C|^2 f_0$.

**Theorem 1.** Let $N$, $n$, $\alpha$ and $\tau$ be fixed and $f(t)$ be a real polynomial such that

$(A0)$ $\deg(f) \leq \tau$;

$(A1)$ $f(t) \leq h(t)$ for $-1 \leq t \leq 1$.

Then $W(N, n, \tau, \alpha) \geq N(f_0 N - f(1))$. 

**Proof.** Let $C \subset S^{n-1}$ be an arbitrary spherical $\tau$-design of $|C| = N$ points. We consecutively have

$$N f(1) + W(C, n, \alpha) = Nf(1) + \sum_{x,y \in C, x \neq y} h(\langle x, y \rangle) \geq |C|f(1) + \sum_{x,y \in C, x \neq y} f(\langle x, y \rangle)$$

$$= |C|^2 f_0 + \sum_{i=1}^{k} \frac{f_i}{r_i} \sum_{j=1}^{r_i} \left( \sum_{x \in C} v_{ij}(x) \right)^2 = N^2 f_0,$$

which implies that $W(C, n, \alpha) \geq N(f_0 N - f(1))$. Since the design $C$ was arbitrary, we conclude that $W(N, n, \tau, \alpha) \geq N(f_0 N - f(1))$. \qed

**Theorem 2.** Let $N$, $n$, $\alpha$ and $\tau$ be fixed and $g(t)$ be a real polynomial such that

$(A0)$ $\deg(g) \leq \tau$;

$(A1')$ $g(t) \geq h(t)$ for $-1 \leq t \leq t_0$, and $g(t) \leq h(t)$ for $t \in [t_0, 1)$ where $t_0$ is such that no $\tau$-design on $S^{n-1}$ of $N$ points can have inner products in the interval $(t_0, 1)$.

Then $W(N, n, \tau, \alpha) \leq N(g_0 N - g(1))$.

**Proof.** Let $C \subset S^{n-1}$ be an arbitrary spherical $\tau$-design of $|C| = N$ points. We consecutively have

$$N g(1) + W(C, n, \alpha) = Ng(1) + \sum_{x,y \in C, x \neq y} h(\langle x, y \rangle) \leq |C|g(1) + \sum_{x,y \in C, x \neq y} g(\langle x, y \rangle)$$

$$= |C|^2 g_0 + \sum_{i=1}^{k} \frac{g_i}{r_i} \sum_{j=1}^{r_i} \left( \sum_{x \in C} v_{ij}(x) \right)^2 = N^2 g_0,$$

which implies that $W(C, n, \alpha) \leq N(g_0 N - g(1))$. Since the design $C$ was arbitrary, we conclude that $W(N, n, \tau, \alpha) \leq N(g_0 N - g(1))$. \qed
We denote by $A_{n, \tau, \alpha}$ ($B_{n, \tau, \alpha}$ resp.) the set of suitable polynomials for Theorem 1 (Theorem 2 resp.). Then Theorems 1 and 2 imply that

$$\sup_{f \in A_{n, \tau, \alpha}} N(f_0 N - f(1)) \leq W(N, n, \tau, \alpha) \leq \inf_{g \in B_{n, \tau, \alpha}} N(g_0 N - g(1)).$$

We show examples with lower and upper bounds for the Riesz energy of 2-designs below.

4 Applications

The next lemma is useful in dealing with the conditions (A1) and (A1').

**Lemma 1.** The equation $f(t) = h(t)$ can not have more than $1 + \deg(f)$ roots (counted with multiplicities).

**Proof.** Let $\deg(f) = k$. The $(k + 1)$-th derivative of the function $h(t) - f(t)$ is $h^{(k+1)}(t) = \frac{\alpha^k (\alpha + 1) \cdots (\alpha + k)}{[2(1-t)]^\alpha + k + 1} > 0$. Then by Rolle’s theorem the $k$-th derivative $h^{(k)}(t) - f^{(k)}(t)$ can have at most one zero and so on, finally the function $h(t) - f(t)$ can have at most $k + 1$ zeros. $\square$

4.1 Lower bounds for 2-designs

We look for a polynomial $f(t) = a_0 + a_1 t + a_2 t^2$ such that $f(-1) = h(-1)$, $f(b) = h(b)$ and $f'(b) = h'(b)$ for some $b \in [-1, 1]$. It follows from Lemma 1 (together with $a_2 > 0$) that $h(t) \geq f(t)$ for every $t \in [-1, 1]$ with equality iff $t = -1$ or $t = b$.

Further, we have

$$f(t) = h(b) + h'(b)(t - b) + \frac{|h'(b)(1 + b) - h(b) + h(-1)(t - b)^2|}{(1 + b)^2}.$$ 

We continue with the function $\Phi_2(b) = Nf_0 - f(1) = \frac{A}{n(1+b)(1-b)}$, where

$$A = N[\alpha h(b)(1+b)(1-nb) - h(b)(1-b)(1-n-2nb) + h(-1)(1-b)(1+nb^2)] - n(1-b)[2\alpha h(b)(1+b) + 4bh(b) + h(-1)(1-b)^2].$$

The equation $\Phi'_2(b) = 0$ gives stationary point $b_2 = -\frac{2n-N}{m(N-2)}$ which appears to set the maximum of $\Phi_2(b)$. We now calculate

$$W(N, n, \alpha, 2) \geq N\Phi_2(b_2) = \frac{2^\alpha n^{\alpha+1}(N-2)^{\alpha+2} + N^{\alpha+1}(N-n-1)(n-1)^\alpha}{4^\alpha N^{\alpha-1}(n-1)^\alpha [N(n+1) - 4n]}$$

(3)
for every $N \geq n + 1$, $n \geq 3$ and $\alpha > 0$. This bound is attained by the so-called bi-orthogonal code which is in fact a 3-design. In this case we have

$$W(2n, n, \alpha, 2) = W(2n, n, \alpha, 3) = N\Phi(b_2) = \frac{2n[(n-1)2^{\alpha+1} + 1]}{4^\alpha}.$$

### 4.2 Upper bounds for 2-designs

Suitable $t_0$ for Theorem 2 are the upper bounds on

$$s(N, \tau) = \max\{s(C) : C \subset \mathbb{S}^{n-1} \text{ is a } \tau\text{-design}, |C| = N\},$$

where $s(C) = \max\{\langle x, y \rangle : x, y \in C, x \neq y\}$. We show such bounds for $\tau = 2$.

**Lemma 2.** [3] We have

$$s(N, 2) \leq \frac{N - 2}{n} - 1.$$

**Proof.** This bound follows from Theorem 3.2 from [3] with suitable polynomial of second degree.

For even $\tau$ and cardinality $N < D(n, \tau + 1)$ we have also lower bounds on

$$\ell(N, \tau) = \min\{\ell(C) : C \subset \mathbb{S}^{n-1} \text{ is a } \tau\text{-design}, |C| = N\},$$

where $\ell(C) = \min\{\langle x, y \rangle : x, y \in C, x \neq y\}$.

**Lemma 3.** [3] We have

$$\ell(N, 2) \geq 1 - \frac{N}{n}.$$

**Proof.** This bound follows from Theorem 3.3 from [3] with suitable polynomial of second degree.

Other bounds on $s(N, \tau)$ and $\ell(N, \tau)$ can be obtained as in Section 4 of [2] with suitable polynomials. For $\tau = 2$ such bounds are worse than these from Lemmas 2 and 3 but become better for larger even $\tau$.

We denote by $s$ and $\ell$ the minimum and the maximum in the right hand side of the estimations of Lemmas 2 and 3, respectively. Then a linear polynomials which graph passes through the points $(\ell, h(\ell))$ and $(s, h(s))$ satisfies the conditions of Theorem 2 and gives the upper bound

$$W(N, n, \alpha, 2) \leq \frac{N[(s(1-s)^\alpha - \ell(1-\ell)^\alpha)(N-1) - (1-\ell)^\alpha + (1-s)^\alpha]}{2^\alpha(1-s)^\alpha(1-\ell)^\alpha(s-\ell)}.$$

This bound can be used, for example, for proving the nonexistence of 2-designs of $n + 2$ points on $\mathbb{S}^{n-1}$ for odd $n$ (see [10]).
References


