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External field energy problems on the sphere and minimal energy points separation

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Opus Citation
External Field Problems on the Sphere and Minimal Energy Points Separation

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* joint work with J. S. Brauchart - TU Graz and E. B. Saff - Vanderbilt
External Field Problem in $\mathbb{C}$ - overview

**Classical energy problems**
- Electrostatics - capacity, equilibrium measures;
- Geometry - transfinite diameter;
- Polynomials - Chebyshev constant discrete orthogonal polynomials
- Classical theorem in potential theory

**External field problems**
- Characterization theorem of weighted equilibrium
- Examples
- Applications to orthogonal polynomials on the real line

**Constrained energy problems**
- Characterization theorem of constrained equilibrium
- Examples
- Applications to discrete orthogonal polynomials
Classical energy problem and equilibrium measure

**Electrostatics - capacity of a conductor** $\text{cap}(E)$

$E$ - compact set in $\mathbb{C}$, $\mu \in \mathcal{M}(E)$ - probability measure on $E$;

Equilibrium occurs when potential (logarithmic) energy $I(\mu)$ is minimized.

$$V_E := \inf \{ I(\mu) := -\int \int \log |x-y| \, d\mu(x)d\mu(y) \}, \quad \text{cap}(E) := \exp(-V_E)$$

**Remark:** For Riesz energy we use Riesz kernel $|x-y|^{-s}$ instead.

**Equilibrium measure** $\mu_E$

If $\text{cap}(E) > 0$, there exists unique $\mu_E$ : $I(\mu_E) = V_E$.

Potential satisfies $U^{\mu_E}(x) = -\int \log |x-y| \, d\mu(y) = C$ on $E$.

**Examples**

- $E = \mathbb{T}$, $d\mu_E = d\theta/(2\pi)$
- $E = [-1, 1]$, $d\mu_E = dx/\pi \sqrt{1-x^2}$
Classical theorem in potential theory

**Geometry - transfinite diameter of a set** $\delta(E)$

$E$ - compact set in $\mathbb{C}$, $Z_n = \{z_1, z_2, \ldots, z_n\} \subset$ of $E$;

Maximize Vandermond (product of all mutual distances)

$$\delta_n(E) := \max_{Z_n \subset E} \left( \prod_{1 \leq i < j \leq n} |z_i - z_j| \right)^{2/(n(n-1))}, \quad \delta(E) := \lim \delta_n(E)$$

**Approximation Theory - Chebyshev constant** $\tau(E)$

$E$ - compact set in $\mathbb{C}$, $T_n(x)$ - monic polynomial of minimal uniform norm;

$$t_n(E) := \min \{ ||x^n - p_{n-1}(x)|| : p_{n-1} \in \mathbb{P}_{n-1} \}, \quad \tau(E) = \lim t_n^{1/n}(E)$$

**Classical theorem (Fekete, Szegö)**

$$\text{cap}(E) = \delta(E) = \tau(E)$$
Electrostatics - add external field

$E$ - closed set in $\mathbb{C}$, $Q$ - lower semi-continuous on $E$ (growth cond.);

$$V_Q := \inf \{I_Q(\mu) := I(\mu) + 2 \int Q(x) \, d\mu(x)\}$$

Theorem - Weighted equilibrium $\mu_Q$

There exists unique $\mu_Q$ : $I_Q(\mu_Q) = V_Q$.

Potential satisfies: $U_{\mu_Q}(x) + Q(x) \geq C$ q.e. on $E$

$U_{\mu_Q}(x) + Q(x) \leq C$ on supp($\mu_Q$).

Applications

- Orthogonal polynomials on real line
- Approximation of functions by weighted polynomials
- Integrable systems
- Random matrices
Proof of the characterization theorem

Let $E$ - compact, $Q$ - continuous. Then $I_Q : \mathcal{M}(E) \to \mathbb{R}$ is lower semi-continuous functional, i.e. if $\mu_n \to \mu$ weak* then

$$\lim \inf I_Q(\mu_n) \geq I_Q(\mu).$$

Let \{\mu_n\} s.t. $I_Q(\mu_n) \to V_Q$. Select a weak* convergent subsequence $\mu_{n_k} \to \mu$, $\mu \in \mathcal{M}(E)$. Then $I_Q(\mu) = V_Q$.

The positive definiteness of the energy functional implies uniqueness.

To show the first characterization inequality, suppose

$$\text{cap}\{x : U^{\mu_Q}(x) + Q(x) < V_Q - \int Q(x) d\mu_Q(z) =: F_Q\} > 0.$$ 

Then there is $n$ s.t. $\text{cap}(K_n) = \text{cap}(\{x : U^{\mu_Q}(x) + Q(x) \leq F_Q - \frac{1}{n}\}) > 0$. Then for small enough $\alpha > 0$, $I_Q(\alpha \mu_{K_n} + (1 - \alpha)\mu_Q) < I_Q(\mu_Q)$.

Finally, if there is $x_0 \in \text{supp}(\mu_Q)$, s.t. $U^{\mu_Q}(x_0) + Q(x_0) > F_Q$ then $I_Q(\mu_Q) > V_Q$, a contradiction.
Constrained energy problem

Electrostatics - add external field and upper constraint

Add constraint measure $\sigma: \sigma(E) > 1$

$$ V^\sigma_Q := \inf \{ I_Q(\mu) := I(\mu) + 2 \int Q(x) d\mu(x) : \mu \leq \sigma \} $$

**Applications:** Discrete orthogonal polynomials, random walks, numerical linear algebra methods, etc.

Theorem (Saff-D. ’97) - Constrained equilibrium $\lambda^\sigma_Q$

There exists unique $\lambda^\sigma_Q : I_Q(\lambda^\sigma_Q) = V^\sigma_Q$.

Potential satisfies:

$U^{\lambda^\sigma_Q}(x) + Q(x) \geq C$ on supp$(\sigma - \lambda^\sigma_Q)$

$U^{\lambda^\sigma_Q}(x) + Q(x) \leq C$ on supp$(\mu)$.

Theorem (Saff-D. ’97) - Constrained vs. weighted equilibrium

If $Q \equiv 0$, then $\sigma - \lambda^\sigma = (\|\sigma\| - 1)\mu_Q$ for $Q(x) = -U^\sigma(x)/(\|\sigma\| - 1)$
Recall from yesterday
Why search for minimal energy optimal) configurations on the sphere?

Numerous applications in:

- Physics
- Biology
- Chemistry
- Computer Science
Thomson Problem (1904) -
("plum pudding" model of an atom)

Find the (most) stable (ground state) energy configuration of \( N \) classical electrons (Coulomb law) constrained to move on the sphere \( S^2 \).

Generalized Thomson Problem (\( 1/r^s \) potentials and \( \log(1/r) \))

A configuration \( \omega_N := \{x_1, \ldots, x_N\} \subset S^2 \) that minimizes Riesz \( s \)-energy

\[
E_s(\omega_N) := \sum_{j \neq k} \frac{1}{|x_j - x_k|^s}, \quad s > 0, \quad E_0(\omega_N) := \sum_{j \neq k} \log \frac{1}{|x_j - x_k|}
\]

is called an optimal \( s \)-energy configuration.
Tammes Problem (1930)
A Dutch botanist that studied modeling of the distribution of the orifices in pollen grain asked the following.

Tammes Problem (Best-Packing)
Place $N$ points on the unit sphere so as to maximize the minimum distance between any pair of points, or, where to situate hostile dictators?
Optimal Configurations in Chemistry

**Fullerenes** (1985) - (Buckyballs)
Vaporizing graphite, Curl, Kroto, Smalley, Heath, and O’Brian discovered $C_{60}$
(Chemistry 1996 Nobel prize)

**Nanotechnology** - Nanowire (R. Smalley)
A giant fullerene molecule few nanometers in diameter, but hundreds of microns (and ultimately meters) in length, with electrical conductivity similar to copper’s, thermal conductivity as high as diamond and tensile strength about 100 times higher than steel.
32 and 122 Electrons and $C_{60}$ and $C_{240}$ Buckyballs
Other "Fullerenes"

Under the lion paw

Montreal biosphere
Computational "Fulerene" - Rob Womersley

1089 Extremal Points on a Sphere
www.maths.unsw.edu.au/~rsw/Sphere
Rob Womersley
UNSW Maths

Sydney VisLab
www.vislab.usyd.edu.au
Visualisation by
Ben Simons

Y-Rotation: 001 degrees

Scaled Cubature Weights
0.65 0.825 1.0 1.175 1.35
Recall: Riesz Optimal Configurations

A configuration \( \omega_N := \{x_1, \ldots, x_N\} \subset S^2 \) that minimizes Riesz \( s \)-energy

\[
E_s(\omega_N) := \sum_{j \neq k} \frac{1}{|x_j - x_k|^s}, \quad s > 0, \quad E_0(\omega_N) := \sum_{j \neq k} \log \frac{1}{|x_j - x_k|}
\]

is called an \textbf{optimal \( s \)-energy configuration}.

- \( s = 0 \), Smale’s problem, logarithmic points (known for \( N = 1 - 6, \ 12 \));
- \( s = 1 \), Thomson Problem (known for \( N = 1 - 6, \ 12 \))
- \( s = -1 \), Fejes-Toth Problem (known for \( N = 1 - 6, \ 12 \))
- \( s \to \infty \), Tammes Problem (known for \( N = 1 - 12, \ 13, \ 14, \ 24 \))
Separation Problem for $\mathbb{S}^d$

**Separation Distance**

$$\delta(\omega_N) := \min_{j \neq k} |x_j - x_k|, \quad \omega_N = \{x_1, \ldots, x_N\}$$

Expect: $\delta(\omega_N^{(s)}) \asymp N^{-1/d}$ as $N \to \infty$, where $\omega_N^{(s)}$ optimal for $\mathbb{S}^d$

**Definition**

A **sequence** of $N$-point configurations $\{\omega_N\}_{N=2}^{\infty} \subset \mathbb{S}^d$ is **well-separated** if there exists some $c > 0$ **not** depending on $N$ s.t.

$$\delta(\omega_N) \geq c N^{-1/d}$$

for all $N$. 
Separation Problem for $\mathbb{S}^d$

<table>
<thead>
<tr>
<th>Condition</th>
<th>Expression</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d = 2, s = 0$</td>
<td>$\delta(\omega_N^{(0)}) \geq \mathcal{O}(N^{-1/2})$</td>
<td>R-S-Z (1995)</td>
</tr>
<tr>
<td>$0 &lt; s &lt; d - 2$</td>
<td>$\delta(\omega_N^{(s)}) \geq ?$</td>
<td></td>
</tr>
<tr>
<td>$s = d - 1$</td>
<td>$\delta(\omega_N^{(d-1)}) \geq \mathcal{O}(N^{-1/d})$</td>
<td>Dahlberg (1978)</td>
</tr>
<tr>
<td>$d - 1 \leq s &lt; d$</td>
<td>$\delta(\omega_N^{(s)}) \geq \mathcal{O}(N^{-1/d})$</td>
<td>K-S-S (2007)</td>
</tr>
<tr>
<td>$d - 2 \leq s &lt; d$</td>
<td>$\delta(\omega_N^{(s)}) \geq \beta_{s,d} N^{-1/d}$</td>
<td>D-S (2007)</td>
</tr>
<tr>
<td>$s = d$</td>
<td>$\delta(\omega_N^{(d)}) \geq \mathcal{O}((N \log N)^{-1/d})$</td>
<td>K-S (1998)</td>
</tr>
<tr>
<td>$s &gt; d$</td>
<td>$\delta(\omega_N^{(s)}) \geq \mathcal{O}(N^{-1/d})$</td>
<td>K-S (1998)</td>
</tr>
<tr>
<td>$s = \infty$</td>
<td>$\delta(\omega_N^{(\infty)}) \geq \mathcal{O}(N^{-1/d})$</td>
<td>Conway-Sloane</td>
</tr>
</tbody>
</table>

Asymptotic Results (H-vdW (1951), Bo-H-S (2007))
Logarithmic Points on $S^2$ $(d = 2, s = 0)$

Separation Results for Logarithmic Configurations on $S^2$

\[ \delta(\omega_N^{(0)}) \geq \frac{3}{5}/\sqrt{N} \]  
R-S-Z (1995)

\[ \delta(\omega_N^{(0)}) \geq \frac{7}{4}/\sqrt{N} \]  
Dubickas (1997)

\[ \delta(\omega_N^{(0)}) \geq \frac{2}{\sqrt{N - 1}} \]  
Dragnev (2002)
Logarithmic Points on $S^2$ ($d = 2$, $s = 0$)

Separation Results for Logarithmic Configurations on $S^2$

$$
\begin{align*}
\delta(\omega_N^{(0)}) & \geq (3/5)/\sqrt{N} & \text{R-S-Z (1995)} \\
\delta(\omega_N^{(0)}) & \geq (7/4)/\sqrt{N} & \text{Dubickas (1997)} \\
\delta(\omega_N^{(0)}) & \geq 2/\sqrt{N - 1} & \text{Dragnev (2002)}
\end{align*}
$$

Proof.

- **R-S-Z, Dubickas**: Stereographical projection with South Pole in $\omega_N$.
- **Dragnev**: Stereographical projection with North Pole in $\omega_N$. This creates external field on projections of remaining $N - 1$ points $\{z_k\}$. **All** weighted Fekete points are contained in support of continuous MEP, i.e. $|z_k| \leq \sqrt{N - 2}$, which implies estimate. $\square$
Separation Problem for $\mathbb{S}^d$ for $d - 2 \leq s < d$

**Approach for $\mathbb{S}^d$**

- Fix a point of $\omega_N^{(s)}$ and consider *external field* $Q_N$ it generates on the remaining $n = N - 1$ points.
- Study continuous energy problem for this external field $Q_N$.
- Discrete energy points for $Q_N$ are *contained* in CEP equilibrium support.

**Theorem (D-Saff 2007)**

$$\delta(\omega_N^{(s,d)}) \geq \frac{K_{s,d}}{N^{1/d}}, \quad K_{s,d} := \left(\frac{2\beta(d/2, 1/2)}{\beta(d/2, (d - s)/2)}\right)^{1/d},$$

where $\beta(x, y)$ denotes the Beta function. In particular,

$$K_{d-1,d} = 2^{1/d}, \quad K_{s,2} = 2\sqrt{1 - s/2}.$$

**Remark:** We need Principle of Domination, de la Valleè-Pousin type theorem, and Riesz balayage, hence the restriction on $s$. 

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[286x266]Peter Dragnev, IPFW
Let $Q$ be an external field. Find $Q$-optimal configuration of $n$ points on $S^d$, that solve

$$\min \left\{ \sum_{j \neq k}^{n} \left[ \frac{1}{|x_j - x_k|^s} + Q(x_j) + Q(x_k) \right] : x_k \in S^d \right\}$$

2007 Separation: $q = 1/(N - 2)$, $R = 1$, $n = N - 1$. 

$Q(x) = \frac{q}{|x - Rp|^s}$
What do **Q-Fekete points** look like?
Example ($S^2$, $s = 1$, $q = 1/3$ and $q = 1$, $n = 4$)
Discrete MEP on $\mathbb{S}^d$ for $d - 2 \leq s < d$

Let $Q$ be an **external field**. Find $Q$-optimal configuration of $n$ points on $\mathbb{S}^d$, that solve

$$
\min \left\{ \sum_{j \neq k}^{n} \left[ \frac{1}{|x_j - x_k|^s} + Q(x_j) + Q(x_k) \right] : x_k \in \mathbb{S}^d \right\}
$$

2007 Separation: $q = 1/(N - 2)$, $R = 1$, $n = N - 1$.

Key idea:
Discrete MEP on $\mathbb{S}^d$ for $d - 2 \leq s < d$

**Q-optimal points**

Let $Q$ be an external field. Find $Q$-optimal configuration of $n$ points on $\mathbb{S}^d$, that solve

$$
\min \left\{ \sum_{j \neq k}^{n} \left[ \frac{1}{|x_j - x_k|^s} + Q(x_j) + Q(x_k) \right] : x_k \in \mathbb{S}^d \right\}
$$

2007 Separation: $q = 1/(N - 2)$, $R = 1$, $n = N - 1$.

**Key idea:**

Theorem

*Q-optimal points are contained in $\text{supp}(\mu_Q)$.***
Example ($\mathbb{S}^2$, $s = 0$, $Q = -\log |x - Rn|$, $20 > R > 1.1$, $n = 1000$)
External field Continuous MEP on $\mathbb{S}^d$ for $d-2 \leq s < d$

$K \subset \mathbb{S}^d$ compact; $\mathcal{M}(K)$ class of positive unit Borel measures $\mu$ supported on $K$

$$U^\mu_s(x) := \int |x - y|^{-s} \, d\mu(y) \quad \mathcal{I}_s[\mu] := \int \int |x - y|^{-s} \, d\mu(x) \, d\mu(y)$$

Riesz $s$-potential of $\mu$ \hspace{1cm} Riesz $s$-energy of $\mu$

$W_s(K) := \inf \{ \mathcal{I}_s[\mu] : \mu \in \mathcal{M}(K) \}$

Riesz $s$-energy of $K$

Extremal measure

Given an external field $Q$ on $K$, there exists unique extremal measure $\mu_Q$ that minimizes the weighted energy

$$\mathcal{I}_s[\mu] + 2 \int Q \, d\mu, \quad \mu \in \mathcal{M}(K),$$

characterized by $U^\mu_s(x) + Q(x) \geq C$ on $\mathbb{S}^d$ with "=" on supp($\mu_Q$).
Physicist’s Problem (Signed Equilibrium)

Given compact $K \subset \mathbb{S}^d$, $Q$ external field on $K$, find a signed measure $\eta_Q$ s.t.

$$U^m_s(x) + Q(x) = \text{const.} \quad \text{everywhere on } K$$

$$\eta_Q(K) = 1$$

**Definition**

$\eta_Q = \eta_{Q,K}$ is called **signed equilibrium on** $K$ **associated with** $Q$.

**Proposition**

If $\eta_Q$ exists, then it is unique.

**Theorem**

Let $\eta_{Q,K} = \eta_{Q,K}^+ - \eta_{Q,K}^-$. Then $\text{supp}(\mu_{Q,K}) \subseteq \text{supp}(\eta_{Q,K}^+)$
Example (Brauchart-Saff-D., 2009)

\[ K = S^d, \quad Q_a(x) = q/|x - a|^s, \quad R = |a| \geq 1 \]

\[ \eta_{Q_a} = \eta_{Q_a}^+ - \eta_{Q_a}^- \]

Let \( \Sigma_t \) be spherical cap centered at South Pole of height \(-1 \leq t \leq 1\)

\[ \text{supp}(\eta_{Q_a}^+) = \Sigma_t(Q_a), \quad \text{supp}(\eta_{Q_a}^-) = S^d \setminus \Sigma_t(Q_a). \]

Remark

If \( \eta_{Q_a} \geq 0 \), then \( \mu_{Q_a} = \eta_{Q_a} \). If not, then \( \text{supp}(\mu_{Q_a}) \subseteq \text{supp}(\eta_{Q_a}^+) \).
Finding $\mu_Q$ when $\text{supp}(\mu_Q) = S^d$

**Gonchar's Problem for $S^d$**

Let $q = 1$, $s = d - 1$ (Newton potential). Find $R_0 > 0$ s.t. for $Q_a(x) = |x - a|^{1-d}$, $a = Rp$

$$\text{supp}(\mu_{Q_a}) \begin{cases} = S^d & \text{if } R \geq R_0, \\ \subset S^d & \text{if } R < R_0. \end{cases}$$

**Proposition**

For $s = d - 1$,

$$d \eta_{Q_a}(x) = \left[1 + \frac{1}{R^{d-1}} - \frac{R^2 - 1}{|x - a|^{d+1}}\right] d \sigma_d(x)$$
Finding $\mu_Q$ when $\text{supp}(\mu_Q) = S^d$

**Gonchar’s Problem for $S^d$**

Let $q = 1$, $s = d - 1$ (Newton potential). Find $R_0 > 0$ s.t. for $Q_a(x) = |x - a|^{1-d}$, $a = Rp$

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For $s = d - 1$,

$$d \eta_{Q_a}(x) = \left[ 1 + \frac{1}{R^{d-1}} - \frac{R^2 - 1}{|x - a|^{d+1}} \right] d \sigma_d(x)$$

If $d = 2$, then $R_0 - 1 = \frac{1 + \sqrt{5}}{2}$. When $d = 4$, $R_0 - 1 = \text{Plastic number}$ from architecture (see Padovan sequence $P_{n+3} = P_{n+1} + P_n$).
Finding $\mu_Q$ when $\text{supp}(\mu_Q) \subsetneq \mathbb{S}^d$; B-D-S (2009)

**Definition ($\mathcal{F}_s$-Mhaskar-Saff functional for general $Q$)**

$$\mathcal{F}_s(K) := W_s(K) + \int Q \, d\mu_K, \quad K \subset \mathbb{S}^d \text{ compact.}$$

**Theorem**

If $d - 2 \leq s < d$ with $s > 0$, then $\mathcal{F}_s$ is minimized for $S_Q := \text{supp}(\mu_Q)$.

**Proposition (Connection to signed equilibrium)**

If $d - 2 < s < d$ with $s > 0$, $Q : K \rightarrow \mathbb{R}$ continuous and $W_s(K) < \infty$, then $U_{s}^{n_Q,\kappa} + Q \equiv \mathcal{F}_s(K)$ on $K$.

**Proof.**

By definition $U_{s}^{n_Q,\kappa}(x) + Q(x) = C$ on $K$. ∎
Finding $\mu_Q$ when $\text{supp}(\mu_Q) \subsetneq \mathbb{S}^d$; B-D-S (2009)

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**Proof.**

\[
\int U^{\eta_Q,K}_s(x) \, d\mu_K(x) + \int Q(x) \, d\mu_K(x) = \int C \, d\mu_K(x)
\]
Definition \((\mathcal{F}_s\text{-Mhaskar-Saff functional for general } Q)\)

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Proposition (Connection to signed equilibrium)

If \(d - 2 < s < d\) with \(s > 0\), \(Q : K \rightarrow \mathbb{R}\) continuous and \(W_s(K) < \infty\), then \(U^\eta_{\mu, K} + Q \equiv \mathcal{F}_s(K)\) on \(K\).

Proof.

\[
\int U^{\mu_K}_s(x) \, d\eta_{Q,K}(x) + \int Q(x) \, d\mu_K(x) = C \int d\mu_K(x)
\]
**Finding \( \mu_Q \) when \( \text{supp}(\mu_Q) \subsetneq S^d \); B-D-S (2009)**

### Definition (\( \mathcal{F}_s \)-Mhaskar-Saff functional for general \( Q \))

\[
\mathcal{F}_s(K) := W_s(K) + \int Q \, d\mu_K, \quad K \subset S^d \text{ compact.}
\]

### Theorem

If \( d - 2 \leq s < d \) with \( s > 0 \), then \( \mathcal{F}_s \) is minimized for \( S_Q := \text{supp}(\mu_Q) \).

### Proposition (Connection to signed equilibrium)

If \( d - 2 < s < d \) with \( s > 0 \), \( Q : K \to \mathbb{R} \) continuous and \( W_s(K) < \infty \), then \( U_{s,\eta}^{\eta_Q, K} + Q \equiv \mathcal{F}_s(K) \) on \( K \).

### Proof.

\[
W_s(K) \int d\eta_{Q,K}(\mathbf{x}) + \int Q(\mathbf{x}) \, d\mu_K(\mathbf{x}) = C \int d\mu_K(\mathbf{x})
\]
Finding $\mu_Q$ when $\text{supp}(\mu_Q) \subsetneq S^d$; B-D-S (2009)

**Definition ($\mathcal{F}_s$-Mhaskar-Saff functional for general $Q$)**

$$\mathcal{F}_s(K) := W_s(K) + \int Q \, d\mu_K, \quad K \subset S^d \text{ compact.}$$

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**Proposition (Connection to signed equilibrium)**

If $d - 2 < s < d$ with $s > 0$, $Q : K \to \mathbb{R}$ continuous and $W_s(K) < \infty$, then $U^{\eta_Q,K}_s + Q \equiv \mathcal{F}_s(K)$ on $K$.

**Proof.**

$$\mathcal{F}_s(K) = W_s(K) + \int Q \, d\mu_K = C$$
Let \( d - 2 \leq s < d, s > 0 \). If \( Q \) is axially symmetric, i.e. \( Q(z) = f(\xi) \), where \( \xi = \text{height of } z \), with \( f \) convex and increasing, then

\[
\text{supp}(\mu_Q) = \Sigma_{t_0} \text{ for some } t_0.
\]

Note: \( Q_a(z) = q/|z - a|^s = f(\xi) \) on \( S^d \) for \( s > 0 \)

Consequently

\[
\text{supp}(\mu_{Q_a}) = \Sigma_{t_0} \text{ for some } t_0.
\]

Theorem (for \( Q_a \))

If \( d - 2 \leq s < d, s > 0 \), and \( a = Rp \), then \( \mathcal{F}_s \) is minimized over \( \Sigma_t \)’s when \( t = t_0 \) is the unique solution of

\[
\frac{W_s(S^d)}{\|\nu_t\|} \left( 1 + q \|\epsilon_t\| \right) = \frac{q(R + 1)^{d-s}}{(R^2 - 2Rt + 1)^{d/2}},
\]

where \( \epsilon_t = \text{Bal}_s(\delta_a, \Sigma_t), \) and \( \nu_t = \text{Bal}_s(\sigma_d, \Sigma_t) \).

or \( t_0 = 1 \) when such a solution does not exist.
The Signed Equilibrium on $\Sigma_t$

**Theorem**

Let $d - 2 < s < d$. $Q_a(x) = q/|x - a|^s$. Signed equilibrium on $\Sigma_t$ is

$$\eta_t \equiv \eta_{Q_a, \Sigma_t} = \frac{1 + q\|\epsilon_t\|}{\|\nu_t\|}\nu_t - q\epsilon_t,$$

$$\epsilon_t = \text{Bal}_s(\delta_a, \Sigma_t), \quad \nu_t = \text{Bal}_s(\sigma_d, \Sigma_t).$$

Moreover,

$$d\eta_t(x) = \eta'_t(u) d\sigma_d(x), \quad x = (\sqrt{1 - u^2}\bar{x}, u) \in \Sigma_t, \quad \bar{x} \in S^{d-1}.$$

The weighted $s$-potential is

$$U^m_s(z) + Q_a(z) = F_s(\Sigma_t) \quad \text{on} \quad \Sigma_t,$$

$$U^m_s(z) + Q_a(z) = F_s(\Sigma_t) + [\cdots] \quad \text{on} \quad S^d \setminus \Sigma_t.$$
Compare with $s = 0$, $d = 2$ case

$t > t_0$,
\[
U_s^{nt}(z) + Q_a(z) \geq F_s(\Sigma_t) \quad \text{on } S^d \setminus \Sigma_t,
\]
\[
U_s^{nt}(z) + Q_a(z) = F_s(\Sigma_t) \quad \text{on } \Sigma_t,
\]
\[
\eta'_t \geq 0 \quad \text{on } \Sigma_t.
\]

$t = t_0$,
\[
U_s^{nt}(z) + Q_a(z) \geq F_s(\Sigma_t) \quad \text{on } S^d \setminus \Sigma_t,
\]
\[
U_s^{nt}(z) + Q_a(z) = F_s(\Sigma_t) \quad \text{on } \Sigma_t,
\]
\[
\eta'_t \not\geq 0 \quad \text{on } \Sigma_t.
\]

$t < t_0$,
\[
U_s^{nt}(z) + Q_a(z) \not\leq F_s(\Sigma_t) \quad \text{on } S^d \setminus \Sigma_t,
\]
\[
U_s^{nt}(z) + Q_a(z) = F_s(\Sigma_t) \quad \text{on } \Sigma_t,
\]
\[
\eta'_t \geq 0 \quad \text{on } \Sigma_t.
\]
Balayage of a measure is superposition of balayages of Dirac-delta’s

**Definition**

Q **positive-axis supported**, if

\[ Q(x) = \int |x - Rp|^{-s} \, d\lambda(R), \quad x \in S^d, \]

for some finite pos. meas. \( \lambda \) supp. on a compact subset of \((0, \infty)\).

**Theorem (Signed equilibrium on \( \Sigma_t \) for positive-axis supported \( Q \))**

Let \( Q \) be as above with \( \text{supp}(\lambda) \subset [1, \infty) \) and \( d - 2 < s < d \). Then

\[ \tilde{\eta}_t = \frac{1 + \|\tilde{\epsilon}_t\|}{\|\nu_t\|} \nu_t - \tilde{\epsilon}_t, \]

where

\[ \nu_t = \text{Bal}_s(\sigma_d, \Sigma_t) \]

\[ \tilde{\epsilon}_t := \text{Bal}_s(\lambda, \Sigma_t) = \int \text{Bal}_s(\delta_Rp, \Sigma_t) \, d\lambda(R) \]
Set \( Q(x) := q/|x - b|^s, |b| > 1 \), let \( \{x_1, x_2, \ldots, x_N\} \) be a \( Q \)-Fekete point set. If \( x_N \) is the fixed, then \( \{x_1, x_2, \ldots, x_{N-1}\} \) is a \( \tilde{Q} \)-Fekete set with \( \tilde{Q}(x) = Q(x) + |x - x_N|^{-s}/(N - 2) \).

**Theorem**

- If \( d - 2 < s < d \), then all \( \tilde{Q} \)-Fekete points are in \( \text{supp}(\mu_{\tilde{Q}}) \).
- In addition, \( \text{supp}(\mu_{\tilde{Q}}) \subseteq \text{supp}(\eta_{\tilde{Q},K}^+) \) for any compact \( \text{supp}(\mu_{\tilde{Q}}) \subseteq K \subseteq S^d \).

\[
\delta(\omega_{Q,N}^{(s)}) \geq \left( \frac{2B(d/2, 1/2)}{(1 + q)B(d/2, (d - s)/2)} \right)^{1/d} N^{-1/d}
\]
Let \( Q(x) := \sum q_i \log \frac{1}{|x - b_i|} \), \( b_i \in S^2 \). (If \( d > 2 \), then \( s = d - 2 \))

**Theorem**

For small enough \( q_i \), the support \( \text{supp}(\mu_Q) \) is found explicitly by removing suitable nonintersecting spherical caps around \( b_i \) and the extremal measure is the normalized surface area measure for \( \text{supp}(\mu_Q) \).
THANK YOU!
Proof.

Suppose $\eta_1$ and $\eta_2$ are two signed $s$-equilibria on $K$. Then

$$U^\eta_1(x) + Q(x) = F_1, \quad U^\eta_2(x) + Q(x) = F_2 \quad \text{for all } x \in K.$$  

Subtracting the two equations and integrating with respect to $\eta_1 - \eta_2$ we obtain

$$\mathcal{I}_s(\eta_1 - \eta_2) = \int [U^\eta_1(x) - U^\eta_2(x)] \, d(\eta_1 - \eta_2)(x) = 0.$$  

We used that $\int (F_2 - F_1) \, d(\eta_1 - \eta_2)(x) = 0$, since $(\eta_1 - \eta_2)(K) = 0$. Therefore $\mathcal{I}_s(\eta) \geq 0$ for any signed measure $\eta$ with equality iff $\eta \equiv 0$.

Therefore $\eta_1 = \eta_2$. \qed