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A Schwarz-Pick lemma for the modulus of holomorphic mappings from the polydisk into the unit ball

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A SCHWARZ-PICK LEMMA FOR THE MODULUS OF HOLOMORPHIC MAPPINGS FROM THE POLYDISK INTO THE UNIT BALL

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Abstract. In this paper we prove a Schwarz-Pick lemma for the modulus of holomorphic mappings from the polydisk into the unit ball. This result extends some related results.

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1. Introduction

Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$, $\mathbb{D}^n$ and $\mathbb{B}_n$ be the polydisk and the unit ball in $\mathbb{C}^n$ respectively. For $z=(z_1,\cdots,z_n)$ and $z'=(z'_1,\cdots,z'_n)\in\mathbb{C}^n$, denote $\langle z,z'\rangle = z_1\overline{z}'_1+\cdots+z_n\overline{z}'_n$ and $|z|=\langle z,z\rangle^{1/2}$. Let $\Omega_{X,Y}$ be the class of all holomorphic mappings f from X into Y, where X is a domain in $\mathbb{C}^n$ and Y is a domain in $\mathbb{C}^m$. For $f\in\Omega_{X,Y}$ and $j=1,\cdots,n$, define

$$|\nabla|f|(z)| = \sup_{\beta\in\mathbb{C}^n,\,|\beta|=1}\left(\lim_{t\in\mathbb{R},\,t\to 0^+}\frac{|f|(z+t\beta)-|f|(z)}{t}\right),\qquad z\in X; \tag{1.1}$$

$$|\nabla_j|f|(z)| = \sup_{\beta\in\mathbb{C},\,|\beta|=1}\left(\lim_{t\in\mathbb{R},\,t\to 0^+}\frac{|f|(z_1,\cdots,z_{j-1},z_j+t\beta,z_{j+1},\cdots,z_n)-|f|(z)}{t}\right),\qquad z\in X, \tag{1.2}$$

where $f=(f_1,\cdots,f_m)$, $|f|=(|f_1|^2+\cdots+|f_m|^2)^{\frac{1}{2}}$ and $z=(z_1,\cdots,z_n)$. Some calculation for $|\nabla|f||$ and $|\nabla_j|f||$ will be given in Section 2.

For $f\in\Omega_{\mathbb{D},\mathbb{D}}$, the classical Schwarz-Pick lemma says that

$$|f'(z)|\le\frac{1-|f(z)|^2}{1-|z|^2},\qquad z\in\mathbb{D}. \tag{1.3}$$

This inequality does not hold for $f\in\Omega_{\mathbb{D},\mathbb{B}_m}$ with $m\ge 2$. For instance, the mapping $f(z)=\frac{1}{\sqrt{2}}(z,1)$ satisfies

$$|f'(0)|=\sqrt{1-|f(0)|^2}>1-|f(0)|^2.$$

However Pavlović [3] found that (1.3) can also be written as

$$|\nabla|f|(z)|\le\frac{1-|f(z)|^2}{1-|z|^2},\qquad z\in\mathbb{D}, \tag{1.4}$$

since (2.3). In [3], Pavlović proved that this form (1.4) can be extended to $\Omega_{\mathbb{D},\mathbb{B}_m}$ and obtained the same inequality for $f\in\Omega_{\mathbb{D},\mathbb{B}_m}$. Recently, we [1] proved that the form (1.4) also can be extended to $\Omega_{\mathbb{B}_n,\mathbb{B}_m}$ and obtained the following inequality for $f\in\Omega_{\mathbb{B}_n,\mathbb{B}_m}$:

$$|\nabla|f|(z)|\le\frac{1-|f(z)|^2}{1-|z|^2},\qquad z\in\mathbb{B}_n. \tag{1.5}$$

In view of the above results, it is interesting for us to consider that if there are some similar results for $f\in\Omega_{\mathbb{D}^n,\mathbb{B}_m}$.

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For $f \in \Omega_{\mathbb{D}^n,\mathbb{D}}$, it is well known [4, 2] that

$$\sum_{j=1}^{n}(1-|z_j|^2)|f'_{z_j}(z)| \le 1-|f(z)|^2 \tag{1.6}$$

for any $z=(z_1,\cdots,z_n)\in\mathbb{D}^n$. This inequality does not hold for $f\in\Omega_{\mathbb{D}^n,\mathbb{B}_m}$ with $m\ge 2$. For instance, the mapping $f(z)=\frac{1}{\sqrt{3}}(z_1,z_2+0.1)\in\Omega_{\mathbb{D}^2,\mathbb{B}_2}$ satisfies

$$\sum_{j=1}^{2}|f'_{z_j}(0)|=\frac{2}{\sqrt{3}}>1-|f(0)|^2.$$

Similarly to (1.4), we find that (1.6) can be written as

$$\sum_{j=1}^{n}(1-|z_j|^2)|\nabla_j|f|(z)| \le 1-|f(z)|^2 \tag{1.7}$$

for any $z=(z_1,\cdots,z_n)\in\mathbb{D}^n$, since (2.5). In view of (1.4) and (1.5), the obvious question is that if the form (1.7) can be extended to $\Omega_{\mathbb{D}^n,\mathbb{B}_m}$ with $m\ge 2$. The following example shows that the form (1.7) can not completely be extended to $\Omega_{\mathbb{D}^n,\mathbb{B}_m}$ with $m\ge 2$: the mapping $f(z)=\frac{1}{\sqrt{2}}(z_1,z_2)\in\Omega_{\mathbb{D}^2,\mathbb{B}_2}$ satisfies

$$\sum_{j=1}^{2}|\nabla_j|f|(0)|=\sqrt{2}>1-|f(0)|^2,$$

since $f(0)=0$ and $|\nabla_j|f|(0)|=|f'_{z_j}(0)|$ for $j=1,2$ by (2.4). However we find that the form (1.7) holds for $f\in\Omega_{\mathbb{D}^n,\mathbb{B}_m}$ at the point $z\in\mathbb{D}^n$ with $f(z)\ne 0$. Precisely:

Theorem 1. *Let $f:\mathbb{D}^n\to\mathbb{B}_m$ be a holomorphic mapping with $m\ge 2$. Then*

$$\sum_{j=1}^{n}(1-|z_j|^2)|\nabla_j|f|(z)| \le 1-|f(z)|^2, \quad \text{if} \quad f(z)\ne 0 \tag{1.8}$$

and

$$\sum_{j=1}^{n}(1-|z_j|^2)^2|\nabla_j|f|(z)|^2 \le 1, \quad \text{if} \quad f(z)=0 \tag{1.9}$$

for any $z=(z_1,\cdots,z_n)\in\mathbb{D}^n$.

The above theorem is the main result in this paper. Note that the inequality in (1.9) always holds whether if $f(z)=0$ or $f(z)\ne 0$. When $f(z)\ne 0$, there is a better inequality, which is (1.8). Theorem 1 is coincident with (1.5) when $n=1$. In addition, (1.8) and (1.9) are sharp. For example, the mapping $f(z)=\frac{1}{\sqrt{2}}\left(\frac{\frac{1}{2}-z_1}{1-\frac{1}{2}z_1},\frac{\frac{1}{2}-z_2}{1-\frac{1}{2}z_2}\right)\in\Omega_{\mathbb{D}^2,\mathbb{B}_2}$ satisfies the equality in (1.8) at $z=0$; the mapping $f(z)=\frac{1}{\sqrt{2}}(z_1,z_2)\in\Omega_{\mathbb{D}^2,\mathbb{B}_2}$ satisfies the equality in (1.9) at $z=0$.

In Section 2, some calculation for $|\nabla|f||$ and $|\nabla_j|f||$ will be given. In Section 3, we will give the proof of Theorem 1 and some remarks for the equality cases in Theorem 1.

2. Some calculation for $|\nabla|f||$ and $|\nabla_j|f||$

For $f\in\Omega_{X,Y}$ with $X\subset\mathbb{C}^n$ and $Y\subset\mathbb{C}^m$, by (1.1) we know that if $|f|(z)\ne 0$ then $|f|$ is $\mathbb{R}$-differentiable at z and $\nabla|f|$ is the ordinary gradient; if $|f|(z)=0$ then $|f|$ is not $\mathbb{R}$-differentiable at z and $\nabla|f|$ is not the ordinary gradient. From Section 2 in [1], we have the following (2.1)-(2.3). For $f\in\Omega_{X,Y}$,

$$|\nabla|f|(z)|=\begin{cases}\frac{1}{|f(z)|}\left|\left(\langle f'_{z_1}(z),f(z)\rangle,\cdots,\langle f'_{z_n}(z),f(z)\rangle\right)\right|, & \text{if} \quad f(z)\ne 0;\\ \sup\limits_{\beta\in\mathbb{C}^n,\,|\beta|=1}|Df(z)\cdot\beta|, & \text{if} \quad f(z)=0,\end{cases} \tag{2.1}$$

where $z=(z_1,\cdots,z_n)\in X$ and $Df(z)\cdot\beta$ is the Fréchet derivative of f at z in the direction β. Then for $f\in\Omega_{X,Y}$ with $X\subset\mathbb{C}$,

$$|\nabla|f|(z)|=\begin{cases}\frac{1}{|f(z)|}\left|\langle f'(z),f(z)\rangle\right|, & \text{if} \quad f(z)\neq 0;\\ |f'(z)|, & \text{if} \quad f(z)=0.\end{cases}\tag{2.2}$$

In particular, for $f\in\Omega_{X,Y}$ with $X\subset\mathbb{C}$ and $Y\subset\mathbb{C}$,

$$|\nabla|f|(z)|=|f'(z)|.\tag{2.3}$$

Then by (1.2) and (2.2), we get that for $f\in\Omega_{X,Y}$ and $j=1,\cdots,n$,

$$|\nabla_j|f|(z)|=\begin{cases}\frac{1}{|f(z)|}\left|\left\langle f'_{z_j}(z),f(z)\right\rangle\right|, & \text{if} \quad f(z)\neq 0;\\ |f'_{z_j}(z)|, & \text{if} \quad f(z)=0,\end{cases}\tag{2.4}$$

where $z=(z_1,\cdots,z_n)\in X$. Note that for the case that $f(z)\neq 0$, if $f'_{z_j}(z)$ and $f(z)$ are collinear, then $|\nabla_j|f|(z)|=|f'_{z_j}(z)|$; if not, then $|\nabla_j|f|(z)|\neq|f'_{z_j}(z)|$. In particular, for $f\in\Omega_{X,Y}$ with $Y\subset\mathbb{C}$,

$$|\nabla_j|f|(z)|=|f'_{z_j}(z)|.\tag{2.5}$$

3. Proof of Theorem 1

First we give one lemma.

Lemma 1. *Let $f(z)=\sum_\alpha a_\alpha z^\alpha\in\Omega_{\mathbb{D}^n,\mathbb{B}_m}$, where $z=(z_1,\cdots,z_n)$, $\alpha=(\alpha_1,\cdots,\alpha_n)$, $z^\alpha=z_1^{\alpha_1},\cdots,z_n^{\alpha_n}$, $f=(f_1,\cdots,f_m)$, $f_j(z)=\sum_\alpha a_{j,\alpha}z^\alpha$ and $a_\alpha=(a_{1,\alpha},\cdots,a_{m,\alpha})$. Then*

$$\sum_\alpha |a_\alpha|^2\leq 1.\tag{3.1}$$

Proof. For $0<\sigma<1$, we have

$$\begin{aligned}1&\geq\frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi}|f(\sigma e^{i\theta_1},\cdots,\sigma e^{i\theta_n})|^2d\theta_1\cdots d\theta_n\\ &=\frac{1}{(2\pi)^n}\sum_{j=1}^m\int_0^{2\pi}\cdots\int_0^{2\pi}|f_j(\sigma e^{i\theta_1},\cdots,\sigma e^{i\theta_n})|^2d\theta_1\cdots d\theta_n\\ &=\sum_{j=1}^m\sum_\alpha|a_{j,\alpha}|^2\sigma^{2|\alpha|}\\ &=\sum_\alpha|a_\alpha|^2\sigma^{2|\alpha|},\end{aligned}$$

where $|\alpha|=\sum_{j=1}^n\alpha_j$. Letting $\sigma\to 1$ gives (3.1). □

Now we give the proof of Theorem 1.

Proof of Theorem 1. First we prove the case that $z=0$.

Therefore we need to prove that

$$\begin{cases}\sum_{j=1}^n|\nabla_j|f|(0)|\leq 1-|f(0)|^2, & \text{if} \quad f(0)\neq 0;\\ \sum_{j=1}^n|\nabla_j|f|(0)|^2\leq 1, & \text{if} \quad f(0)=0.\end{cases}\tag{3.2}$$

By (2.4), it suffices to prove that

$$\sum_{j=1}^n\left|\left\langle f'_{z_j}(0),\frac{f(0)}{|f(0)|}\right\rangle\right|\leq 1-|f(0)|^2, \quad \text{if} \quad f(0)\neq 0\tag{3.3}$$

and

$$\sum_{j=1}^{n} |f'_{z_j}(0)|^2 \le 1, \quad \text{if} \quad f(0) = 0. \tag{3.4}$$

Obviously, (3.4) holds by Lemma 1. For (3.3), let

$$h(z) = \left\langle f(z), \frac{f(0)}{|f(0)|} \right\rangle, \qquad z \in \mathbb{D}^n.$$

Then $h(z)$ is a holomorphic function from $\mathbb{D}^n$ into $\mathbb{D}$, $h(0) = |f(0)|$, and for $j = 1, \cdots, n$,

$$h'_{z_j}(0) = \left\langle f'_{z_j}(0), \frac{f(0)}{|f(0)|} \right\rangle, \tag{3.5}$$

where $z = (z_1, \cdots, z_n)$. Applying (1.6) to h and by (3.5) we get

$$\begin{aligned}\sum_{j=1}^{n} \left| \left\langle f'_{z_j}(0), \frac{f(0)}{|f(0)|} \right\rangle \right| &= \sum_{j=1}^{n} |h'_{z_j}(0)| \\ &\le 1 - |h(0)|^2 \\ &= 1 - |f(0)|^2.\end{aligned}$$

Then (3.3) is proved. Therefore (3.2) is proved.

Now we prove the case that $z = p \neq 0$.

Let $p = (p_1, \cdots, p_n)$ and

$$g(w) = f(\varphi(w)), \qquad w = (w_1, \cdots, w_n) \in \mathbb{D}^n,$$

where $\varphi(w) = (\varphi_1(w_1), \cdots, \varphi_n(w_n))$, $\varphi_j(w_j) = \frac{p_j - w_j}{1 - \overline{p_j} w_j}$ for $j = 1, \cdots, n$. Then $g(w)$ is a holomorphic mapping from $\mathbb{D}^n$ into $\mathbb{B}_m$, $g(0) = f(p)$, and for $j = 1, \cdots, n$,

$$g'_{w_j}(0) = f'_{z_j}(p)(-1 + |p_j|^2). \tag{3.6}$$

For the case that $f(p) \neq 0$, applying (3.3) to g and by (2.4), (3.6) we get

$$\begin{aligned}\sum_{j=1}^{n} (1 - |p_j|^2) |\nabla_j |f|(p)| &= \sum_{j=1}^{n} (1 - |p_j|^2) \left| \left\langle f'_{z_j}(p), \frac{f(p)}{|f(p)|} \right\rangle \right| \\ &= \sum_{j=1}^{n} \left| \left\langle g'_{w_j}(0), \frac{g(0)}{|g(0)|} \right\rangle \right| \\ &\le 1 - |g(0)|^2 \\ &= 1 - |f(p)|^2.\end{aligned}$$

For the case that $f(p) = 0$, applying (3.4) to g and by (2.4), (3.6) we get

$$\begin{aligned}\sum_{j=1}^{n} (1 - |p_j|^2)^2 |\nabla_j |f|(p)|^2 &= \sum_{j=1}^{n} (1 - |p_j|^2)^2 |f'_{z_j}(p)|^2 \\ &= \sum_{j=1}^{n} |g'_{w_j}(0)|^2 \\ &\le 1.\end{aligned}$$

Then the theorem is proved. □

In the following, we give some remarks for the equality cases in Theorem 1.

Remark 1. When $n = 1$, (1.8) and (1.9) reduce to (1.5). The equality case in (1.5) has been discussed in [1].

Remark 2. When $n \geq 2$, if the equality in (1.9) holds at some point $p = (p_1, \cdots, p_n)$, then the structure of the expression of f will be controlled. Precisely:

$$f(z) = \sum_{j=1}^{n} f'_{z_j}(p)(-1 + |p_j|^2)\frac{p_j - z_j}{1 - \overline{p_j} z_j}, \qquad z \in \mathbb{D}^n,$$

which is obvious by the proof of Theorem 1, Lemma 1 and (2.4).

Remark 3. When $n \geq 2$, if the equality in (1.8) holds at some point $p = (p_1, \cdots, p_n)$, then the following discussion shows that the equality at p is not enough to control the structure of the expression of f. By the proof of Theorem 1, we know that the key to the extremal problem of (1.8) at the point p is to solve the extremal problem of (1.6) at $z = 0$. That is: for $h \in \Omega_{\mathbb{D}^n,\mathbb{D}}$, if $\sum_{j=1}^{n} |h'_{z_j}(0)| = 1 - |h(0)|^2$, then what the structure of the expression of h is. By the proof of (1.6) in [4], we only need to consider this problem: for $h \in \Omega_{\mathbb{D}^n,\mathbb{D}}$ with $h(0) = 0$, if $\sum_{j=1}^{n} |h'_{z_j}(0)| = 1$, then what the structure of the expression of h is. However, the following examples show that the condition $\sum_{j=1}^{n} |h'_{z_j}(0)| = 1$ can not control the higher order terms in the expansion of h. Consequently, the structure of the expression of h can not be controlled.
Examples:

$$g(z) = \frac{1}{2}z_1 + \frac{1}{2}z_2 \in \Omega_{\mathbb{D}^2,\mathbb{D}};$$

$$\tilde{g}(z) = \frac{\frac{1}{2}z_1 + \frac{1}{2}z_2 - z_1 z_2}{1 - \frac{1}{2}z_1 - \frac{1}{2}z_2} \in \Omega_{\mathbb{D}^2,\mathbb{D}}.$$

Although the above two functions satisfy $g(0) = \tilde{g}(0) = 0$, $g'_{z_1}(0) = \tilde{g}'_{z_1}(0)$, $g'_{z_2}(0) = \tilde{g}'_{z_2}(0)$ and $\sum_{j=1}^{n} |g'_{z_j}(0)| = \sum_{j=1}^{n} |\tilde{g}'_{z_j}(0)| = 1$, the expression of g has no higher order terms and the expression of $\tilde{g}$ has some higher order terms.

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